

Geometry I, 2006 : Coursework 2 Solutions

2. [1 mark each if everything is correct; some calculation must be shown]

(i) Let $\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. Then $|\underline{r}|^2 = 2^2 + 1^2 + 1^2 = 6$, and so $|r| = \sqrt{6}$. Thus $\hat{r} = \frac{1}{\sqrt{6}}\underline{r} = \begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$
(which can also be written as $\frac{2}{\sqrt{6}}\underline{i} - \frac{1}{\sqrt{6}}\underline{j} + \frac{1}{\sqrt{6}}\underline{k}$).

(ii) Let $\underline{r} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$. Then $|\underline{r}|^2 = 3^2 + 0^2 + 2^2 = 13$, and so $|r| = \sqrt{13}$. Thus $\hat{r} = \frac{1}{\sqrt{13}}\underline{r} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ 0 \\ \frac{-2}{\sqrt{13}} \end{pmatrix}$ (which can also be written as $\frac{3}{\sqrt{13}}\underline{i} - \frac{2}{\sqrt{13}}\underline{k}$).

3. (i) [1 mark] Let $\underline{p}$ be the position vector of the point $P = (2, -1, 3)$, and let $\underline{u} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$. Then the vector equation for the line through P in the direction of $\underline{u}$ is $\underline{r} = \underline{p} + \lambda\underline{u}$; that is,

$$\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}.$$

[1 mark] This gives the parametric equations:

$$\begin{cases} x &= 2 &- \lambda \\ y &= -1 &+ 2\lambda \\ z &= 3 &+ 4\lambda \end{cases}.$$

[1 mark] Eliminating λ gives the Cartesian equations:

$$\frac{x-2}{-1} = \frac{y+1}{2} = \frac{z-3}{4}.$$

(ii) [1 mark; no mark for simple yes or no answer] The point $(0, 3, 10)$ is not on this line, since it does not satisfy, for example, the parametric equations. If it did, then we would have $0 = 2 - \lambda$, and so $\lambda = 2$, but then $10 \neq 3 + 4\lambda = 11$.

4. (i) [1 mark; some calculation must be shown]

Let $P = (-2, 5, 4)$ and $Q = (2, 5, 11)$, let $\underline{p}$ be the position vector of P , and let $\underline{u}$ be the vector represented by $\overrightarrow{PQ}$. Then the line through points P and Q is the line through P in the direction of $\underline{u} = \underline{q} - \underline{p} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix}$. The vector equation for this line is $\underline{r} = \underline{p} + \lambda\underline{u}$; that is,

$$\underline{r} = \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix}.$$

[1 mark] This gives the parametric equations:

$$\begin{cases} x &= -2 &+& 4\lambda \\ y &= 5 \\ z &= 4 &+& 7\lambda \end{cases}.$$

[1 mark] Eliminating λ gives the Cartesian equations:

$$y = 5, \frac{x+2}{4} = \frac{z-4}{7}.$$

(ii) [1 mark; no mark for simple yes or no answer] The point $(-6, 5, 3)$ is not on this line, since it does not satisfy, for example, the Cartesian equations, as $\frac{-6+2}{4} = -1 \neq \frac{3-4}{7}$.

(iii) [1 mark; no mark for simple yes or no answer] The point $(-6, 5, -3)$ is on this line, since it satisfies, for example, the vector equation. We have

$$\begin{pmatrix} -6 \\ 5 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix}.$$

5. (i) [The answer is in the lecture notes.]

[1 mark] The *scalar product* $\underline{u} \cdot \underline{v}$ is $|\underline{u}||\underline{v}| \cos \theta$, if $\underline{u}, \underline{v}$ are non-zero vectors and θ is the angle between them. If $\underline{u} = \underline{0}$ or $\underline{v} = \underline{0}$ we define $\underline{u} \cdot \underline{v} = 0$.

[1 mark] We say that $\underline{u}$ and $\underline{v}$ are *orthogonal* if $\underline{u} \cdot \underline{v} = 0$.

(ii) [1 mark] (One possibility.) The non-zero vector $\underline{u} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ is orthogonal to $\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$,

since their scalar product is 0.

(iii) [2 marks for correct proof; 1 mark if one error; no marks otherwise] Suppose that $\underline{u} \cdot \underline{v} = 0$ for every vector $\underline{v}$. Then, in particular, $0 = \underline{u} \cdot \underline{u} = |\underline{u}|^2$, and so $|\underline{u}| = 0$, so $\underline{u} = \underline{0}$.

6. (i) [2 marks for correct proof; 1 mark for reasonable progress]

If $\underline{u}$ is parallel to $\underline{v}$ then they are in the same or opposite directions and so the angle θ between $\underline{u}$ and $\underline{v}$ must be 0 or π . In both cases, $|\cos \theta| = 1$. Now by definition, $\underline{u} \cdot \underline{v} = |\underline{u}||\underline{v}| \cos \theta$, and so $|\underline{u} \cdot \underline{v}| = |\underline{u}||\underline{v}| |\cos \theta| = |\underline{u}||\underline{v}|$ (as $|\cos \theta| = 1$).

(ii) [2 marks for correct proof; 1 mark for reasonable progress]

If $\underline{u}$ is not parallel to $\underline{v}$ then they are neither in the same nor opposite directions and so the angle θ between $\underline{u}$ and $\underline{v}$ must satisfy $0 < \theta < \pi$, and so $|\cos \theta| < 1$. Now by definition, $\underline{u} \cdot \underline{v} = |\underline{u}||\underline{v}| \cos \theta$, and so $|\underline{u} \cdot \underline{v}| = |\underline{u}||\underline{v}| |\cos \theta| < |\underline{u}||\underline{v}|$ (as $|\cos \theta| < 1$, and $|\underline{u}||\underline{v}| \neq 0$ since $\underline{u}, \underline{v} \neq \underline{0}$).